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Standing Stokes waves of maximum height

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An analytic expression is found for an infinite subset of the coefficients of the perturbation expansion. They are the coefficients of the terms most rapidly varying at each order, which are also the first terms in the expansion of each Fourier coefficient. The sum of these terms gives a nonlinear approximation to the solution. At greatest height this approximation has a profile with a 90° corner.

1. Introduction

Although Stokes waves have been discussed for a long time, it has usually been the progressing wave that has been considered. The standing wave, by contrast, has received little attention. No accepted proof exists showing that the largest standing wave must have a corner of some fixed angle, nor any approximation technique based upon an expansion about such a corner. All that remains is the small amplitude expansion. This is an unreliable tool for obtaining behaviour at large amplitude. Penney & Price (1952) computed this expansion to five terms, and from this predicted a 90° corner. Taylor (1953) performed experiments and concluded that this was approximately correct, although he doubted their reasoning. Edge & Walters (1964) later made similar experiments, and found that the angle at the corner varied near 90°.

The solution for the wave height γ can be written in the form

$$\gamma = \sum_{n=0}^{\infty} \gamma_n \cos(nkx), \tag{1.1}$$

where
$$\gamma_n = \gamma_n(t) = \sum_{m=0}^{\infty} \gamma_{nm} \epsilon^{n+2m}$$

is of order ϵ^n when ϵ is small. It is assumed that there is no problem with resonance.

By considering the partial sum that contains only the γ_{n0} terms, it is possible to solve for γ_{n0} for all n. (In fact this operation is carried out in a transformed frame of reference.) These terms are the first in the expansions of each harmonic amplitude γ_n. They are also the terms most rapidly varying at each order of the expansion in powers of ϵ. The explicit result is

$$\gamma_{n0} = n^{n-1} 2^{1-n} \cos^n t/n!. \tag{1.2}$$

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This analytic result for an infinite subset of the expansion coefficients is new. At the very least, it provides a useful check on numerical work. By the same technique, a similar result can be found for the progressing wave, and was found by Wilton (1914) directly from the recurrence relation.

The sum γ_0 of all these terms gives an approximation to the standing-wave solution. It represents a wave with a 90° corner at greatest height.

2. The equations of motion

Take units of length and time such that the gravitational acceleration is unity and the wavelength of linearized waves equals 2π. The equations of irrotational motion of a heavy inviscid fluid are, at the surface $y = \gamma(x,t)$, the Bernoulli condition

$$\phi_t + y + \tfrac{1}{2}(\nabla\phi)^2 = 0 \tag{2.1}$$

and the kinematic condition

$$\gamma_t + \phi_x\gamma_x - \phi_y = 0. \tag{2.2}$$

Below the surface, ϕ is harmonic, and at great depth velocities vanish. An arbitrary function of time, in the Bernoulli condition, has been absorbed into ϕ. We refer to x, y space as 'physical space'.

These equations contain the wave height γ as one of the unknowns, and so are transcendentally nonlinear. This can be improved by an analytic transformation of the fluid volume into a fixed half-space. The equations have 'only' a polynomial nonlinearity, and manipulations become easier.

Let

$$z = x + iy = Z(\zeta), \quad \zeta = \xi + i\eta, \tag{2.3}$$

with $Z \sim \zeta$ at great depth and Z so chosen that the surface $y = \gamma(x,t)$ is given by $\eta = 0$.

The transformation must be analytic and invertible inside the fluid:

$$dZ/d\zeta \neq 0, \quad \eta \leqslant 0. \tag{2.4}$$

We shall consider only waves symmetric about the crest. This crest can be chosen to be at $\xi = 0$. Then either both, or neither, of (x,y) and $(-x,y)$ are on the surface. This implies that

$$Z^*(\zeta) = -Z(-\zeta), \tag{2.5}$$

where * denotes a complex conjugate.

We now find the equations of motion in the new frame of reference. Let X and Y be the real and imaginary parts of Z,

$$Z(\zeta,t) = X(\xi,\eta,t) + iY(\xi,\eta,t), \tag{2.6}$$

and

$$\phi = \Phi(\xi,\eta,t). \tag{2.7}$$

The transformation is analytic, so

$$\nabla_\xi^2\Phi = 0, \quad \eta \leqslant 0; \tag{2.8}$$

also

$$\nabla_\xi\Phi \to 0 \quad \text{as} \quad \eta \to -\infty. \tag{2.9}$$

∇_ξ denotes the gradient operator in ξ, η space.

Now $$Z(\zeta,t) = z, \quad Z_t + Z_\zeta \zeta_t = 0, \quad \zeta_t = -Z_t/Z_\zeta,$$

and taking real and imaginary parts yields

$$\xi_t = (X_\eta Y_t - Y_\eta X_t)/D, \quad \eta_t = (Y_\xi X_t - X_\xi Y_t)/D, \quad D = X_\xi Y_\eta - Y_\xi X_\eta = |dZ/d\zeta|^2. \tag{2.10}$$

Then for the Bernoulli condition

$$0 = \phi_t + y + \tfrac{1}{2}(\nabla\phi)^2 = Y + \Phi_t + \Phi_\xi \xi_t + \Phi_\eta \eta_t + \tfrac{1}{2}(\nabla_\zeta \Phi)^2 |d\zeta/dZ|^2,$$

$$D(\Phi_t + Y) + \Phi_\xi(X_\eta Y_t - Y_\eta X_t) + \Phi_\eta(Y_\xi X_t - X_\xi Y_t) + \tfrac{1}{2}(\nabla_\zeta \Phi)^2 = 0. \tag{2.1$'$}$$

The kinematic condition states that a particle on the surface remains on it. That is,

$$0 = \frac{D}{Dt}(\eta) = \eta_t + \nabla_\phi \nabla_\eta = \frac{Y_\xi X_t - X_\xi Y_t}{D} + \left|\frac{d\zeta}{dZ}\right|^2 (\nabla_\zeta \Phi \nabla_\zeta \eta),$$

$$Y_\xi X_t - X_\xi Y_t + \Phi_Y = 0. \tag{2.2$'$}$$

We now put $W = \Phi + i\Psi$, the complex potential, and $Z = \zeta + iF$. Expressing the two surface conditions in terms of W and F and their conjugates gives

$$(1 - iF'^*)\{(1 + iF')(F + W_t) - iW'F_t\} + \tfrac{1}{2}W'W'^* + \text{c.c.} = 0 \quad \text{(Bernoulli)}, \tag{2.1$''$}$$

$$iW' - F_t(1 - iF'^*) + \text{c.c.} = 0 \quad \text{(kinematic)}, \tag{2.2$''$}$$

where a prime denotes $\partial/\partial\zeta$. Both of these apply at $\eta = 0$. Inside the fluid, both W and f are analytic and bounded functions of ζ.

3. The approximation technique

The linearized form of the surface conditions (2.1$''$) and (2.2$''$) is

$$W_t + F = 0, \quad iW' - F_t = 0,$$

with the solution

$$F = \epsilon \cos t e^{-i\zeta}, \quad W = -\epsilon \sin t e^{-i\zeta}. \tag{3.1}$$

These are the first terms in an expansion in powers of the amplitude ϵ (the Stokes expansion):

$$\left.\begin{aligned} F &= \sum_{n=0}^{\infty} f_n \theta^n, \quad W = \sum_{n=0}^{\infty} w_n \theta^n, \\ f_n &= f_n(t) = \sum_{m=0}^{\infty} f_{nm} \epsilon^{n+2m}, \quad w_n = w_n(t) = \sum_{n=0}^{\infty} w_{nm} \epsilon^{n+2m}, \\ \theta &= e^{-ik\zeta}, \quad k = \sum_{m=0}^{\infty} k_m \epsilon^{2m}, \\ k_0 &= 1, \quad f_{00} = w_{00} = 0, \quad f_{10} = \cos t, \quad w_{10} = -\sin t. \end{aligned}\right\} \tag{3.2}$$

f_n and w_n, the nth Fourier coefficients, are of order ϵ^n when ϵ is small. The f_{nm} and w_{nm} are given recursively by substitution of this expansion into the equations of motion. The symmetry condition implies that all the f_{nm} and w_{nm} are real. This form of expansion assumes that there is no problem with resonance.

When these expansions are substituted into the surface conditions (2.1″) and (2.2″), there result expressions of the form

$$\sum_{n,m=0}^{\infty} T_{nm}\epsilon^{n+2m}(\theta^n+\theta^{*n}) = 0, \tag{3.3}$$

and the T_{nm} depend on the f_{nm} and w_{nm} and their time derivatives. The equations for the f_{nm} and w_{nm} are just $T_{nm} = 0$.

Now the T_{n0} depend only upon the f_{n0} and w_{n0}, since it is only these terms that contain expressions $\epsilon^{n+2n}\theta^n$ with $n = 0$, or the power of ϵ equal to the power of θ.

Consider the effect of multiplying an analytic and a conjugate term. The powers of ϵ of the two will add, but their frequencies subtract. Thus terms with $m > 0$ only can be so generated.

Thus, if we omit all conjugate terms from the boundary conditions, we have

$$\sum_{n,m=0}^{\infty} T'_{nm}\epsilon^{n+2m}\theta^n = 0 \tag{3.4}$$

but $T'_{n0} = T_{n0}$, i.e. the dependence of T_{n0} upon the f_{n0} and w_{n0} is unchanged. Consequently, by solving the simpler (because analytic) surface condition (3.4), we can find f_{n0} and w_{n0} for all n.

So let F_0 and W_0 be defined by

$$\left.\begin{aligned} F_0 &= \sum_{n=0}^{\infty} f_{n0}(t)\,\epsilon^n\theta^n \quad (f_{00}=0),\\ W_0 &= \sum_{n=1}^{\infty} w_{n0}(t)\,\epsilon^n\theta^n \quad (w_{00}=0),\\ k &= k_0 = 1. \end{aligned}\right\} \tag{3.5}$$

These satisfy the surface conditions (2.1″) and (2.2″) with the conjugate terms omitted:

$$iW_0' - \partial F_0/\partial t = 0, \tag{3.6}$$

$$(1+iF_0')\left(\frac{\partial W_0}{\partial t}+F_0\right) - iW_0'\frac{\partial F_0}{\partial t} = 0. \tag{3.7}$$

F_0 and W_0 are the first terms in a systematic expansion, which is more fully described in appendix A. We shall in fact only solve for F_0 and W_0, and so it is important to see what they are. This initial approximation can be viewed in two ways.

The first is that it approximates the Fourier coefficients f_n and w_n by the first terms $f_{n0}\epsilon^n$ and $w_{n0}\epsilon^n$ of their expansions in powers of ϵ. This is done for all n.

The second interpretation is more interesting. The expansion (3.2) can also be written as

$$F = \sum_{n=1}^{\infty} \epsilon^n a_n(\theta), \quad W = \sum_{n=1}^{\infty} \epsilon^n b_n(\theta),$$

where a_n and b_n are polynomials of degree n. F_0 and W_0 contain the θ^n term in a_n and b_n: the highest power of θ at each order of ϵ. Restated, F_0 and W_0 contain, at each order of ϵ, those terms most rapidly varying in space. No approximation is made on the time dependence.

4. Solution of the approximate equations

One of the major reasons for using this method is that (3.6) and (3.7) have an exact solution:

$$F_0 = H(\mu), \tag{4.1}$$

$$W_0 = -\tan tH(\mu), \tag{4.2}$$

$$\mu = \epsilon \cos t\, e^{-i\zeta}, \tag{4.3}$$

where
$$\mu H_\mu - H(1+\mu H_\mu) = 0; \quad H \sim \mu \quad \text{as} \quad \mu \to 0. \tag{4.4}$$

This can be checked by direct substitution. Rearranging (4.4),

$$d\mu/\mu = (1/H - 1)\,dH, \tag{4.5}$$

$$\ln \mu = \ln H - H + \text{constant},$$

$$\mu = He^{-H}. \tag{4.6}$$

By Lagrange's formula
$$H = \sum_{n=0}^{\infty} d_n \mu^n,$$

$$d_n = \frac{1}{n!}\left[\frac{d^{n-1}}{dH^{n-1}}(e^H)^n\right]_{H=0} = \frac{n^{n-1}}{n!},$$

$$H = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \epsilon^n \cos^n t\, e^{-in\zeta}, \tag{4.7}$$

and hence
$$f_{n0} = \frac{n^{n-1}}{n!} \cos^n t \quad \text{for all } n > 0. \tag{4.8}$$

From (4.5), H has a maximum value of 1. This is attained when $\mu = e^{-1}$, and so $\epsilon = e^{-1}$ is the radius of convergence of the series (4.7). Near this singularity, i.e. for H^{-1} small,

$$\mu = He^{-H} = [1+(H^{-1})]\,e^{-1}[1-(H^{-1})+\tfrac{1}{2}(H^{-1})^2]\ldots$$

$$= e^{-1}[1-\tfrac{1}{2}(H^{-1})^2+\ldots],$$

$$H = 1-[2(1-e\mu)]^{\frac{1}{2}}+\ldots \quad (\mu \text{ near } e^{-1}). \tag{4.9}$$

H has a square-root singularity at $\mu = e^{-1}$. This corresponds to a surface with a $90°$ corner, as is shown below.

Expressions corresponding to these can be found in physical space. There too there is an expansion in powers of ϵ, and the terms most rapidly varying at each order are given by substituting F_0 into the transformation

$$z = \zeta + iF_0 = \zeta + iH(\epsilon e^{-i\zeta}\cos t),$$

$$i(\zeta - z) = H(\epsilon \cos t\, e^{-i\zeta}) = \epsilon e^{-i\zeta} \cos t\, e^{-H}$$

$$= \epsilon e^{-i\zeta} \cos t\, e^{i(\zeta - z)} = \epsilon e^{-iz}\cos t. \tag{4.10}$$

On the surface, $\zeta = \xi =$ real. Taking real parts of (4.10) gives

$$y = \epsilon e^{y} \cos t \cos x,$$

$$y e^{-y} = \epsilon \cos t \cos x,$$

$$y = H(\epsilon \cos t \cos x). \tag{4.11}$$

This describes a wave that is smooth for $\epsilon < e^{-1}$. When $\epsilon = e^{-1}$, there is a $90°$ corner at $\cos t \cos x = 1$. Suppose that $\epsilon = e^{-1}$ and x and t are small, then

$$y = H\{e^{-1}[1 - \tfrac{1}{2}(t^2 + x^2)]\}$$

$$= 1 - (t^2 + x^2)^{\frac{1}{2}} + \dots \quad \text{by} \quad (4.9), \tag{4.12}$$

so that at $t = 0$, $y = 1 - |x|$, a $90°$ corner.

This wave is smooth everywhere, except that at maximum amplitude it attains instantaneously a profile with a $90°$ corner.

Corresponding to (4.10) is an expression for the potential:

$$W_0 = -\tan t F_0 = -\epsilon e^{-iz} \sin t$$

or in real terms

$$\phi_0 = -\epsilon \sin t \cos x\, e^{y} \tag{4.13}$$

compared with

$$F_0 = \epsilon \cos t\, e^{-iz}. \tag{4.14}$$

Equations (4.13) and (4.14) can be derived by transforming (3.6) and (3.7) into equations with z and t as dependent variables. Let

$$W(\zeta, t) = V(z, t) \quad \text{and} \quad f(\zeta, t) = h(z, t),$$

then

$$i\frac{\partial V_0}{\partial z} - \frac{\partial h_0}{\partial t} = 0, \quad \frac{\partial V_0}{\partial t} + h_0 = 0, \tag{4.15}$$

with solutions (4.13) and (4.14). It should be emphasized that (4.13) and (4.14) are not linear approximations. They contain all the most rapidly varying terms. The fact that they stop at $O(\epsilon)$ asserts that these terms vanish at all higher orders, i.e., that at order ϵ^2 there are no e^{-2iz} terms, and so on. By comparison, the expression (4.11) for the surface elevation does contain terms of all orders. Equations (4.15) can also be derived directly, without the use of the ζ plane. But the nonlinearity is so awkward, in this formulation, that it is essentially impossible to go any further, to do the analysis that comes below.

Equation (4.11) gives, for the free surface,

$$y = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \epsilon \cos^n t \cos^n x$$

$$= \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \epsilon^n \cos^n t\, (2^{1-n} \cos nx + \text{lower harmonics}). \tag{4.16}$$

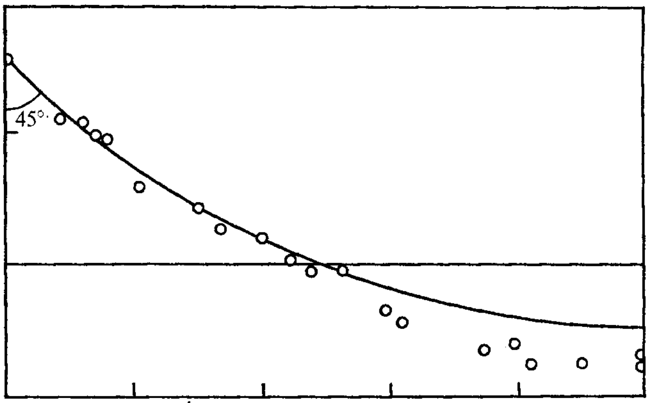

FIGURE 1. Lowest approximation, wave of greatest height: $y = H(e^{-1}\cos x)$. Circles represent points from Taylor's experiments, plotted with crests coinciding.

Hence the coefficient of $\cos nx$, at order ϵ^n, is

$$\gamma_{n0} = 2^{1-n}\frac{n^{n-1}}{n!}\cos^n t. \tag{4.17}$$

This result, giving an infinite subset of the expansion coefficients, is new. A similar result for the progressing wave,

$$f_{n0} = n^{n-1}/n!, \quad \text{corresponding to } y = F(\epsilon\cos x),$$

can be found by the same technique. The expression for the f_{n0} was found by Wilton (1914) from the recurrence relation, although he did not find the expression for their sum.

The expression (4.17) shows that Penney & Price made a mistake at fifth order. They had, at lowest order,

$$y = A\sin\sigma t\cos x.$$

So, in their notation, the correct coefficient of $\cos 5x$, to order A^5, is

$$2^{-4}\frac{5^4}{5!}(A\sin\sigma t)^5 = A^5\frac{125}{6144}(\sin 5\sigma t - 5\sin 3\sigma t + 10\sin\sigma t). \tag{4.18}$$

Penney & Price's calculations were checked by Tadjbakhsh & Keller (1960) as far as third order and found to be correct.

Taylor (1953) performed experiments that produced waves with 90° corners. These fitted Penney & Price's results well except, naturally, at the corner. Edge & Walters (1964) also performed experiments and found that the angle was near 90°, but was not fixed. They obtained some waves with angles of 85°.

Taylor's experimental profiles have values of H/L = height/wavelength of 0·22–0·24. The profile corresponding to our lowest approximation has

$$H/L = 0{\cdot}203.$$

Figure 1 above shows this profile, with some of Taylor's results plotted. The shapes are similar, with the 10–20 % difference in height. Higher-order approximations, while they would not alter the 90° corner, would of course change the shape of the wave somewhat.

One defect of this expansion is that desirable physical properties, such as that particles on the surface stay on it, or that volume is conserved, are satisfied at each order of the expansion not exactly, but only to that order. Thus the curve in figure 1 has more volume in the crest than has been taken out of the trough: F_0 conserves volume only to order ϵ. The rationale of the expansion is purely mathematical, and does not have a physical basis.

5. Behaviour near the corner

Despite the fact that F_0 gives the correct corner angle, more careful analysis shows that it is not a good approximation in the vicinity of the corner.

For $\epsilon = e^{-1}$ and t and ζ small

$$F_0 = H(\epsilon \cos t\, e^{-i\zeta}) = H[e^{-1}(1-\tfrac{1}{2}t^2 - i\zeta)]$$
$$= 1-[t^2+2i\zeta]^{\frac{1}{2}}\dots \tag{5.1}$$

Thus F_0 has a similarity scale ζ/t^2 near the corner. But this is not the correct scale.

It is easier to work in terms of $W(\zeta, t)$ and $Z(\zeta, t)$ rather than W and F. Consider a wave of maximum amplitude. At time $t = 0$, it has a corner at $\zeta = 0$. As t approaches zero, assume an inner scale contracting with time and an inner solution, namely

$$Z = t^r\tilde{Z}(\tilde{\zeta}), \quad W = t^q\tilde{W}(\tilde{\zeta}), \quad \zeta = t^s\tilde{\zeta}. \tag{5.2}$$

At time $t = 0$, the surface has a corner of angle, say, $\pi\alpha$. Then Z has a singularity of order α: $Z \sim \zeta^\alpha$ as $\zeta \to 0$. The inner solution must match to this as $\tilde{\zeta} \to \infty$:

$$\tilde{Z} \sim \tilde{\zeta}^\alpha \quad \text{as} \quad \tilde{\zeta} \to \infty, \tag{5.3}$$

then

$$Z \sim t^r(\zeta t^{-s})^\alpha,$$

so

$$r = \alpha s. \tag{5.4}$$

Equation (5.3) is the boundary condition at infinity on $\tilde{Z}$. Similarly, with the potential, if

$$\tilde{W} \sim \tilde{\zeta}^{\alpha'} \quad \text{as} \quad \tilde{\zeta} \to \infty$$

then

$$W \sim t^{q-\alpha' s}\zeta^{\alpha'}.$$

At time $t = 0$ the wave attains its maximum elevation, and is instantaneously at rest. The velocity potential should vanish, so

$$q = s\alpha' + 1. \tag{5.5}$$

The boundary conditions at $y = 0$ are

$$Z'Z'^*(W_t - iZ) + W'Z_t(-Z'^*) + \tfrac{1}{2}W'W'^* + \text{c.c.} = 0, \tag{5.6}$$

$$iW' + iZ_t(-Z'^*) + \text{c.c.} = 0, \tag{5.7}$$

with both Z and W analytic below. The boundedness condition at infinity is lost in this local analysis. It could only be applied to an outer solution and influence Z and W through matching.

Derivatives transform appropriately:

$$\frac{\partial}{\partial \zeta} \to t^{-s}\frac{\partial}{\partial \tilde{\zeta}}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - s\tilde{\zeta}t^{-1}\frac{\partial}{\partial \tilde{\zeta}}.$$

Using a prime here for $\partial/\partial\tilde{\zeta}$, the surface conditions are, at $\tilde{\eta} = 0$,

$$\tilde{Z}'\tilde{Z}'^*[q\tilde{W} + t^{r-q+1}(-i\tilde{Z})] + r\tilde{W}'\tilde{Z}(-\tilde{Z}'^*) + \tfrac{1}{2}\tilde{W}'\tilde{W}'^* + \text{c.c.} = 0, \tag{5.8}$$

$$t^{q-2r+1}i\tilde{W}' + ir\tilde{Z}(-\tilde{Z}'^*) + \text{c.c.} = 0, \tag{5.9}$$

with $\tilde{W}$ and $\tilde{Z}$ analytic below. A non-trivial balance requires

$$q = 2r-1, \quad q = r+1.$$

The four equations for r, q, s, α and α' then give

$$r = 2, \quad q = 3, \quad \alpha = \alpha', \quad s\alpha = 2. \tag{5.10}$$

Unfortunately, neither the corner angle $\pi\alpha$ nor the similarity scale ζt^{-s} is determined. One constraint $s\alpha = 2$ relates them. For a 90° corner, $\alpha = \frac{1}{2}$, and the similarity scale at the corner is ζt^{-4}, rather than the scale ζt^{-2} of f_0.

The inner problem defined by (5.8) and (5.9) does not have a simple solution. It is, however, fairly easy to solve it in the outer limit $\tilde{\zeta} \to \infty$, and thus obtain some constraints any matching outer solution must satisfy.

Let

$$\tilde{W} \sim \gamma_0(i\tilde{\zeta})^{\alpha}, \tag{5.11}$$

$$\tilde{Z} \sim i\sigma_0(i\tilde{\zeta})^{\alpha} + i\sigma_1 \tag{5.12}$$

as $\tilde{\zeta} \to \infty$, where γ_0, σ_0 and σ_1 are real constants. In the kinematic equation (5.9), the term $\tilde{Z}(-\tilde{Z}'^*) + \text{c.c.}$ vanishes at highest order, and it is necessary to have two terms in the expansion (5.12). Inserting this in (5.9) gives

$$\gamma_0 = 2\sigma_0\sigma_1. \tag{5.13}$$

The fact that the highest-order term here vanishes means that the term $Z_t(-Z'^*)$ in (5.7) is not as singular as it appears. The leading singular term $i\sigma(i\tilde{\zeta})^{\alpha}$ in $\tilde{Z}$ interacts not with itself but with the considerably more innocuous term σ_1, a constant. Written in ζ space, the expansion (5.12) is $Z = i\sigma_0(i\zeta)^{\alpha} + i\sigma_1 t^2 + \ldots$ as $\zeta, t \to 0$ with $\zeta t^{-s} \to \infty$. The singular corner term $i\sigma_0(i\zeta)^{\alpha}$ interacts at highest order with just the simple function of time $i\alpha t^2$. This is a consequence of the fact that Z enters (5.7) in the form (time derivatives of Z) (space derivatives of Z). The first term $i\sigma_0(i\zeta)^{\alpha}$ in this expansion represents the corner at $t = 0$, and is independent of time. This is important when it comes to constructing a better approximation.

The Bernoulli condition (5.8) simplifies to

$$\gamma_0 = -\sigma_1, \quad \text{so} \quad \sigma_1 = -\tfrac{1}{2}. \tag{5.14}$$

Equation (5.14) implies that the acceleration of the surface and fluid near the corner is -1, or in dimensional terms, $-g$. This is hardly a surprise.

For the surface

$$z = t^2\tilde{Z}(\tilde{\zeta}) = i\sigma_0(i\xi)^{\alpha} - \tfrac{1}{2}it^2,$$

or $$x = -\sigma_0 \xi^\alpha \sin \tfrac{1}{2}\pi\alpha, \quad y = \sigma_0 \xi^\alpha \cos \tfrac{1}{2}\pi\alpha - \tfrac{1}{2}t^2,$$

so $$d^2y/dt^2 = -1.$$

For the flow field: $$W \sim t^3\gamma_0(i\tilde{\zeta})^\alpha \sim -t\sigma_0(i\zeta)^\alpha \sim itz.$$

The corresponding velocities are

$$\text{horizontal velocity} = 0, \quad \text{vertical velocity} = t.$$

These results are independent of α.

At $t = 0$, the fluid is stationary with unit acceleration downwards. The flow consists of a straight up and down motion. There is at highest order no singularity in the flow field. This contrasts with the progressing wave, where the singular behaviour of the flow forces the 120° angle. The corner is a streamline of the progressing wave, and the flow must consequently be singular. The corner shape for the standing wave is the locus of the highest points attained by particles on the surface of the fluid. This locus can be singular, while the particle paths are considerably better behaved. It is probably because of this that the similarity analysis does not determine the corner angle. The weak nature of the flow near the corner probably also implies that the shape of the corner can be easily disturbed. This would explain the results of Edge & Walters.

It is possible to construct, by a rather involved technique with similarities to PLK, an expansion where the first term has a 90° corner and the correct similarity scale. This is outlined in appendix B, and given in full in Grant (1972).

I am considerably indebted to Professor D. J. Benney for his help and guidance.

Appendix A

F_0 and W_0 depend on ϵ and θ only in the combination $\xi = \epsilon\theta$. Higher terms in the expansion are found by expressing everything in terms of ξ and ϵ and expanding in powers of ϵ^2, holding ξ constant:

$$F = \sum_{n,m=0}^{\infty} f_{nm}\epsilon^{n+2m}\theta^n = \sum_{n=0}^{\infty} \epsilon^{2n}F_n(\xi), \tag{A 1}$$

$$F_n(\xi) = \sum_{m=0}^{\infty} f_{nm}(\epsilon\theta)^n, \quad k = \sum_{m=0}^{\infty} k_m\epsilon^{2m}. \tag{A 2), (A 3}$$

This gives the expansion of an analytic term. For a conjugate term

$$f_{nm}\epsilon^{n+2m}\theta^{*n} = f_{nm}\epsilon^{2n+2m}\xi^{-n}. \tag{A 4}$$

For example, to evaluate the kinematic condition to order ϵ^2 we have

$$iW' = k\xi W\xi = (1+k_1\epsilon^2)(\xi W_{0\xi} + \epsilon^2\xi W_{1\xi}),$$

$$F_t = F_{0t} + \epsilon^2 F_{1t}, \quad -iF'^* = \epsilon^2 f_{10}\xi^{-1},$$

$$F_t^* = \epsilon^2 f_{10t}\xi^{-1} + \epsilon^2 f_{01t},$$

$$
\begin{aligned}
0 &= iW' - iW'^* - F_t(1 - iF'^*) - F_t^*(1 + iF') \\
&= (1 + k_1\epsilon^2)(\xi W_{0\xi} + \epsilon^2 \xi W_{1\xi}) + \epsilon^2 w_{10}\xi^{-1} - (F_{0t} + \epsilon^2 F_{1t})(1 + \epsilon^2 f_{10}\xi^{-1}) \\
&\qquad - \epsilon^2(f_{10t}\xi^{-1} + f_{01t})(1 + \xi F_{0\xi}) \\
&= (\xi W_{0\xi} - F_{0t}) + \epsilon^2\{k_1 \xi W_{0\xi} + w_{10}\xi^{-1} - F_{0t} f_{10}\xi^{-1} - (f_{10t}\xi^{-1} + f_{01t})(1 + \xi F_{0\xi}) \\
&\qquad + (\xi W_{1\xi} - F_{1t})\}. \qquad \text{(A 5)}
\end{aligned}
$$

k_1 is determined by the requirement that the solution be analytic in ξ. This equation contains ξ^{-1} terms, and at order ϵ^{2n} there will be terms up to ξ^{-n}. All of these cancel automatically, and so the equations are analytic in the dependent variables ϵ and ξ.

Appendix B. Outline of a better approximation

Section 5 showed that F_0 cannot be a good approximation near the corner, for it does not even have the correct similarity scale there. The reason F_0 fails near the corner is that the approximation defining it is a bad one for the kinematic equation

$$iW' - F_t(1 - iF'^*) + \text{c.c.} = 0, \qquad \text{(B 1)}$$

while

$$iW_0' - F_{0t} + \text{c.c.} = 0. \qquad \text{(B 2)}$$

Near the corner, $F_0' \sim \zeta^{-\frac{1}{2}}$, and the omitted term dominates those retained.

A better approximation can be defined by

$${}^1F_0 = H(\lambda), \quad {}^1W_0 = -(\gamma(t)/\beta(t))\,H(\lambda), \quad \lambda = \epsilon\beta(t)\,\theta. \qquad \text{(B 3)}$$

The profile at maximum amplitude is unchanged, but it is approached at a different rate. At maximum amplitude, near the corner

$$\lambda \sim 1 - \tfrac{1}{4}t^4$$

and so

$${}^1F_0 \sim 1 - (2i\zeta + \tfrac{1}{2}t^4)^{\frac{1}{2}}, \qquad \text{(B 4)}$$

showing the correct similarity scale.

Instead of (B 2), ${}^1F_0$ and ${}^1W_0$ satisfy

$$i^1W_0' - {}^1F_{0t}[1 + S(-i^1F_0'^*)] = 0, \qquad \text{(B 5)}$$

where S is defined by

$$S(G) = G(\epsilon^2/\beta^l\epsilon_{\max}, t, \epsilon) \quad \text{given} \quad G(\theta, t, \epsilon). \qquad \text{(B 6)}$$

$S(G)$ is a function of time only, and represents G evaluated near the corner. The reasons for this choice are as follows.

(i) There are exact solutions for ${}^1F_0$ and ${}^1W_0$.

(ii) ${}^1F_0$ and ${}^1W_0$ have the correct similarity scale near the corner.

(iii) In fact, ${}^1F_0$ and ${}^1W_0$ are correct to highest order in the matching limit near the corner approaching the corner with the similarity variable ζt^{-4} large.

(iv) It is suggested by a PLK-type argument to make succeeding terms in the expansion less singular. Following the expansion as in appendix A, the omitted term $[{}^1F_{0t}(-i^1F_0'^*)]$ in (B 2) generates a forcing term, at order ϵ^{2n},

$$J_n = \epsilon^{2n}\beta^n\lambda^{-n\,1}f_{n0t}(-i^1F_0'^*). \qquad \text{(B 7)}$$

This is singular at $\lambda = \lambda_{\max} = e^{-1}$, and is the most singular forcing term. We cancel this singularity by modifying the forcing term to

$$J_n' = \epsilon^{2n}\beta^n(\lambda^{-n} - e^n)^1 f_{n0t}(-i^1 F_0'^*). \tag{B8}$$

Adding to the basic equation the negative of all the terms so added gives the expression (B5). This is described in full detail in Grant (1972). The most important result is that it modifies F_0 only by replacing $\cos t$ by $\beta(t) = \cos t + O(\epsilon^2)$, and that the profiles are not altered, but only the speed at which the wave passes through them.

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